



## **sup x inf inequality on manifold of dimension 3**

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# SUP-INF INEQUALITY ON MANIFOLD OF DIMENSION 3

SAMY SKANDER BAHOURA

**ABSTRACT.** We give an estimate of type  $\sup \times \inf$  on Riemannian manifold of Dimension 3 for prescribed curvature equation.

## INTRODUCTION AND RESULTS.

We are on Riemannian manifold  $(M, g)$  of dimension 3 not necessary compact. In this paper we denote  $\Delta = -\nabla^j(\nabla_j)$  the geometric laplacian.

The scalar curvature equation is:

$$8\Delta u + R_g u = V u^{N-1}, \quad u > 0. \quad (E)$$

Where  $R_g$  is the scalar curvature and  $V$  is a function (prescribed scalar curvature).

We consider three positive real number  $a, b, A$  and we suppose  $V$  lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } \|\nabla V\|_{L^\infty(M)} \leq A. \quad (C)$$

The equation  $(E)$  was studied lot of when  $M = \Omega \subset \mathbb{R}^n$  or  $M = \mathbb{S}_n$  see for example, [B], [CL1], [L1]. In this case we have a  $\sup \times \inf$  inequality.

The corresponding equation in two dimensions on open set  $\Omega$  of  $\mathbb{R}^2$ , is:

$$\Delta u = V e^u, \quad (E')$$

The equation  $(E')$  was studied lot of and we can find very important result about a priori estimates in [BM], [BLS], [CL2], [L2], and [S].

In the case  $V \equiv 1$  and  $M$  compact, the equation  $(E)$  is Yamabe equation. It was studied lot of, T.Aubin and R.Schoen have proved the existence of solution in this cas, see for example [Au] and [L-P].

When  $M$  is a compact Riemannian manifold, it exist some compactness result for equation  $(E)$  see [L-Zh]. Li and Zhu [L-Zh], proved that the energy is bounded and if we suppose  $M$  not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose  $M$  Riemannian manifold (not necessarily compact) and  $V \equiv 1$ , Li and Zhang [L-Z] proved that the product  $\sup \times \inf$  is bounded.

Here, we give an equality of type  $\sup \times \inf$  for the equation  $(E)$  with general conditions  $(C)$ . We have:

*Theorem.* For all compact set  $K$  of  $M$  and all positive numbers  $a, b, A$ , it exists a positive constant  $c$ , which depends only on,  $K, a, b, A, M, g$  such that:

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all  $u$  solution of  $(E)$  with conditions  $(C)$ .

Note that in our work, we have not assumption on energy or boundary condition if we suppose the manifold  $M$  with boundary.

Next, in the proof of the previous theorem, we can replace the scalar curvature by any smooth function  $f$ , but here we do the proof with  $R_g$  the scalar curvature.

### Proof of the theorem.

#### Part I: The metric and the laplacian in polar coordinates.

Let  $(M, g)$  a Riemannian manifold. We note  $g_{x,ij}$  the local expression of the metric  $g$  in the exponential map centred in  $x$ .

We are concerning by the polar coordinates expression of the metric. By using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{ij}^k(r, \theta) d\theta^i d\theta^j = dt^2 + r^2 \tilde{g}_{ij}^k(r, \theta) d\theta^i d\theta^j = g_{x,ij} dx^i dx^j,$$

in a polar chart with origin  $x$ ,  $]0, \epsilon_0[ \times U^k$ , with  $(U^k, \psi)$  a chart of  $\mathbb{S}_{n-1}$ . We can write the element volume:

$$dV_g = r^{n-1} \sqrt{|\tilde{g}^k|} dr d\theta^1 \dots d\theta^{n-1} = \sqrt{[\det(g_{x,ij})]} dx^1 \dots dx^n,$$

then,

$$dV_g = r^{n-1} \sqrt{[\det(g_{x,ij})]} [\exp_x(r\theta)] \alpha^k(\theta) dr d\theta^1 \dots d\theta^{n-1},$$

where,  $\alpha^k$  is such that,  $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta) d\theta^1 \dots d\theta^{n-1}$ . (Riemannian volume element of the sphere in the chart  $(U^k, \psi)$ ).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta) \sqrt{[\det(g_{x,ij})]},$$

Clearly, we have the following proposition:

**Proposition 1:** Let  $x_0 \in M$ , there exist  $\epsilon_1 > 0$  and if we reduce  $U^k$ , we have:

$$|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

and,

$$|\partial_r |\tilde{g}^k|(x, r, \theta)| + \partial_r \partial_{\theta^m} |\tilde{g}^k|(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

**Remark:**

$\partial_r [\log \sqrt{|\tilde{g}^k|}]$  is a local function of  $\theta$ , and the restriction of the global function on the sphere  $\mathbb{S}_{n-1}$ ,  $\partial_r [\log \sqrt{\det(g_{x,ij})}]$ . We will note,  $J(x, r, \theta) = \sqrt{\det(g_{x,ij})}$ .

Let's write the laplacian in  $[0, \epsilon_1] \times U^k$ ,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log \sqrt{|\tilde{g}^k|}] \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r \log J(x, r, \theta) \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We write the laplacian ( radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log J(x, r, \theta)] \partial_r - \Delta_{S_r(x)},$$

where  $\Delta_{S_r(x)}$  is the laplacian on the sphere  $S_r(x)$ .

We set  $L_\theta(x, r)(\dots) = r^2 \Delta_{S_r(x)}(\dots) [\exp_x(r\theta)]$ , clearly, this operator is a laplacian on  $\mathbb{S}_{n-1}$  for particular metric. We write,

$$L_\theta(x, r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [J(x, r, \theta)] \partial_r - \frac{1}{r^2} L_\theta(x, r).$$

If,  $u$  is function on  $M$ , then,  $\bar{u}(r, \theta) = u \circ \exp_x(r\theta)$  is the corresponding function in polar coordinates centred in  $x$ . We have,

$$-\Delta u = \partial_{rr} \bar{u} + \frac{n-1}{r} \partial_r \bar{u} + \partial_r [J(x, r, \theta)] \partial_r \bar{u} - \frac{1}{r^2} L_\theta(x, r) \bar{u}.$$

## **Part II: "Blow-up" and "Moving-plane" methods**

### **The "blow-up" technic**

Let,  $(u_i)_i$  a sequence of functions on  $M$  such that,

$$\Delta u_i + R_g u_i = V_i u_i^5, \quad u_i > 0, \quad (E')$$

We argue by contradiction and we suppose that  $\sup^{1/3} \times \inf$  is not bounded.

We assume that:

$\forall c, R > 0 \exists u_{c,R}$  solution of  $(E')$  such that:

$$R \left[ \sup_{B(x_0, R)} u_{c,R} \right]^{1/3} \times \inf_M u_{c,R} \geq c, \quad (H)$$

### **Proposition 2:**

There exist a sequence of points  $(y_i)_i$ ,  $y_i \rightarrow x_0$  and two sequences of positive real number  $(l_i)_i, (L_i)_i$ ,  $l_i \rightarrow 0$ ,  $L_i \rightarrow +\infty$ , such that if we consider  $v_i(y) = \frac{u_i \circ \exp_{y_i}(y)}{u_i(y_i)}$ , we have:

$$0 < v_i(y) \leq \beta_i \leq 2^{1/2}, \quad \beta_i \rightarrow 1.$$

$$v_i(y) \rightarrow \left( \frac{1}{1 + |y|^2} \right)^{1/2}, \quad \text{uniformly on every compact set of } \mathbb{R}^3.$$

$$l_i^{1/2} [u_i(y_i)]^{1/3} \times \inf_M u_i \rightarrow +\infty$$

### **Proof:**

We use the hypothesis  $(H)$ , we can take two sequences  $R_i > 0$ ,  $R_i \rightarrow 0$  and  $c_i \rightarrow +\infty$ , such that,

$$R_i \left[ \sup_{B(x_0, R_i)} u_i \right]^{1/3} \times \inf_M u_i \geq c_i \rightarrow +\infty,$$

Let,  $x_i \in B(x_0, R_i)$ , such that  $\sup_{B(x_0, R_i)} u_i = u_i(x_i)$  and  $s_i(x) = [R_i - d(x, x_i)]^{1/2} u_i(x)$ ,  $x \in B(x_i, R_i)$ . Then,  $x_i \rightarrow x_0$ .

We have,

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{1/2} u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

Set :

$$l_i = R_i - d(y_i, x_i), \quad \bar{u}_i(y) = u_i \circ \exp_{y_i}(y), \quad v_i(z) = \frac{u_i \circ \exp_{y_i}(z / [u_i(y_i)]^2)}{u_i(y_i)}.$$

Clearly,  $y_i \rightarrow x_0$ . We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2}} [u_i(y_i)]^2 = \frac{[s_i(y_i)]^2}{c_i^{1/2}} \geq \frac{c_i^1}{c_i^{1/2}} = c_i^{1/2} \rightarrow +\infty.$$

If  $|z| \leq L_i$ , then  $y = \exp_{y_i}[z/[u_i(y_i)]^2] \in B(y_i, \delta_i l_i)$  with  $\delta_i = \frac{1}{(c_i)^{1/2}}$  and  $d(y, y_i) < R_i - d(y_i, x_i)$ , thus,  $d(y, x_i) < R_i$  and,  $s_i(y) \leq s_i(y_i)$ , we can write,

$$u_i(y)[R_i - d(y, y_i)]^{1/2} \leq u_i(y_i)(l_i)^{1/2}.$$

But,  $d(y, y_i) \leq \delta_i l_i$ ,  $R_i > l_i$  and  $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$ , we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[ \frac{l_i}{l_i(1 - \delta_i)} \right]^{1/2} \leq 2^{1/2}.$$

We set,  $\beta_i = \left( \frac{1}{1 - \delta_i} \right)^{1/2}$ , clearly  $\beta_i \rightarrow 1$ .

The function  $v_i$  is solution of:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk}v_i - \partial_k \left[ g^{jk} \sqrt{|g|} \right] [\exp_{y_i}(y)]\partial_j v_i + \frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^4} v_i = V_i v_i^5,$$

By elliptic estimates and Ascoli, Ladyzenskaya theorems,  $(v_i)_i$  converge uniformly on each compact to the function  $v$  solution on  $\mathbb{R}^3$  of,

$$\Delta v = V(x_0)v^5, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2^{1/2},$$

Without loss of generality, we can suppose  $V(x_0) = 3$ .

By using maximum principle, we have  $v > 0$  on  $\mathbb{R}^3$ , the result of Caffarelli-Gidas-Spruck (see [C-G-S]) give,  $v(y) = \left( \frac{1}{1 + |y|^2} \right)^{1/2}$ . We have the same properties for  $v_i$  in the previous paper [B2].

### **Polar coordinates and "moving-plane" method**

Let,

$$w_i(t, \theta) = e^{1/2} \bar{u}_i(e^t, \theta) = e^{t/2} u_i \circ \exp_{y_i}(e^t \theta), \quad \text{et } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

### **Lemma 1:**

The function  $w_i$  is solution of:

$$-\partial_{tt}w_i - \partial_t a \partial_t w_i - L_\theta(y_i, e^t) + c w_i = V_i w_i^5,$$

with,

$$c = c(y_i, t, \theta) = \left( \frac{1}{2} \right)^2 + \frac{1}{2} \partial_t a - \lambda e^{2t},$$

### **Proof:**

We write:

$$\partial_t w_i = e^{3t/2} \partial_r \bar{u}_i + \frac{1}{2} w_i, \quad \partial_{tt} w_i = e^{5t/2} \left[ \partial_{rr} \bar{u}_i + \frac{2}{e^t} \partial_r \bar{u}_i \right] + \left( \frac{1}{2} \right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \quad \partial_t a \partial_t w_i = e^{5t/2} [\partial_r \log J \partial_r \bar{u}_i] + \frac{1}{2} \partial_t a w_i.$$

the lemma is proved.

Now we have,  $\partial_t a = \frac{\partial_t b_1}{b_1}$ ,  $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$ ,

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt}(\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = V_i w_i^{N-1},$$

where,  $b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2$ .

Let,

$$\tilde{w}_i = \sqrt{b_1}w_i.$$

**Lemma 2:**

The function  $\tilde{w}_i$  is solution of:

$$\begin{aligned} -\partial_{tt}\tilde{w}_i + \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)\tilde{w}_i = \\ = V_i \left( \frac{1}{b_1} \right)^2 \tilde{w}_i^5, \end{aligned}$$

where,  $c_2 = [\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$ .

**Proof:**

We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i + (c + b_2)\tilde{w}_i = V_i \left( \frac{1}{b_1} \right)^2 \tilde{w}_i^5,$$

But,

$$\Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}w_i) = \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i - 2\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} + w_i \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i) = w_i \nabla_\theta \sqrt{b_1} + \sqrt{b_1} \nabla_\theta w_i,$$

we deduce,

$$\sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i = \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - c_2\tilde{w}_i,$$

with  $c_2 = [\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$ . The lemma is proved.

**The "moving-plane" method:**

Let  $\xi_i$  a real number, and suppose  $\xi_i \leq t$ , we set  $t^{\xi_i} = 2\xi_i - t$  and  $\tilde{w}_i^{\xi_i}(t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta)$ .

We have,

$$\begin{aligned} -\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_{g_{y_i, e^{t^{\xi_i}}, \mathbb{S}_2}}(\tilde{w}_i^{\xi_i}) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}) \cdot \nabla_\theta \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}, \cdot)b_2(t^{\xi_i}) - c_2^{\xi_i}]\tilde{w}_i^{\xi_i} = \\ = V_i^{\xi_i} \left( \frac{1}{b_1^{\xi_i}} \right)^2 (\tilde{w}_i^{\xi_i})^5. \end{aligned}$$

By using the same arguments than in [B], we have:

**Proposition 3:**

We have:

$$1) \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \forall \theta \in \mathbb{S}_2.$$

For all  $\beta > 0$ , there exist  $c_\beta > 0$  such than:

$$2) \frac{1}{c_\beta}e^{t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{t/2}, \forall t \leq \beta, \forall \theta \in \mathbb{S}_2.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(\dots) + \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\dots) + 2\nabla_\theta(\dots) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\dots)$$

**Remark:** In the operator  $\bar{Z}_i$ , by using the proposition 3, the coefficient  $c + b_1^{-1/2}b_2 - c_2$  verify:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \text{ pour } t \ll 0,$$

it is fundamental if we want to apply Hopf maximum principle.

**Goal:**

Like in [B], we have elliptic second order operator, here it's  $\bar{Z}_i$ , the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0.$$

We write:

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &= (\Delta_{g_{y_i, e^t \xi_i, \mathbb{S}_2}} - \Delta_{g_{y_i, e^t, \mathbb{S}_2}})(\tilde{w}_i^{\xi_i}) + \\ &+ 2(\nabla_{\theta, e^t \xi_i} - \nabla_{\theta, e^t})(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^t \xi_i} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta, e^t}(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^t \xi_i} [\log(\sqrt{b_1^{\xi_i}}) - \log \sqrt{b_1}] + \\ &+ 2\nabla_{\theta, e^t} \tilde{w}_i^{\xi_i} \cdot (\nabla_{\theta, e^t \xi_i} - \nabla_{\theta, e^t}) \log \sqrt{b_1} - [(c + b_1^{-1/2}b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2}b_2 - c_2)] \tilde{w}_i^{\xi_i} + \\ &+ V_i^{\xi_i} \left( \frac{1}{b_1^{\xi_i}} \right)^2 (\tilde{w}_i^{\xi_i})^5 - V_i \left( \frac{1}{b_1} \right)^2 \tilde{w}_i^5. \quad (***) \end{aligned}$$

Clearly, we have:

**Lemma 3 :**

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} \text{Ricci}_{y_i}(\theta, \theta) e^{2t} + \dots,$$

$$R_g(e^t \theta) = R_g(y_i) + \langle \nabla R_g(y_i) | \theta \rangle e^t + \dots$$

According to proposition 1 and lemma 3,

**Proposition 4 :**

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &\leq V_i b_1^{-2} [(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + 2(\tilde{w}_i^{\xi_i})^5 |V_i^{\xi_i} - V_i| + \\ &+ C |e^{2t} - e^{2t \xi_i}| \left[ |\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})| + |\text{Ricci}_{y_i}| [|\tilde{w}_i^{\xi_i}| + (\tilde{w}_i^{\xi_i})^5] + |R_g(y_i)| \tilde{w}_i^{\xi_i} \right] + C' |e^{3t \xi_i} - e^{3t}|. \end{aligned}$$

**Proof:**

We use proposition 1, we have:

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log b_1, |\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq C e^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j, \theta_k} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C e^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i}) - \partial_t b_1(t)| \leq C' |e^{2t} - e^{2t \xi_i}|, \text{ sur } ]-\infty, \log \epsilon_1] \times \mathbb{S}_2, \forall x \in B(x_0, \epsilon_1)$$

Locally,

$$\Delta_{g_{y_i, e^t, \mathbb{S}_2}} = L_\theta(y_i, e^t) = -\frac{1}{\sqrt{|\tilde{g}^k(e^t, \theta)|}} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k(e^t, \theta)|} \partial_{\theta^j}].$$

Thus, in  $[0, \epsilon_1] \times U^k$ , we have,

$$A_i = \left[ \left[ \frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}) \right]^{\xi_i} - \frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}) \right] (\tilde{w}_i^{\xi_i})$$

then,  $A_i = B_i + D_i$  with,

$$B_i = \left[ \tilde{g}^{\theta^l \theta^j} (e^{t^{\xi_i}}, \theta) - \tilde{g}^{\theta^l \theta^j} (e^t, \theta) \right] \partial_{\theta^l \theta^j} \tilde{w}_i^{\xi_i}(t, \theta),$$

and,

$$D_i = \left[ \frac{1}{\sqrt{|\tilde{g}^k|(e^{t^{\xi_i}}, \theta)}} \partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j} (e^{t^{\xi_i}}, \theta) \sqrt{|\tilde{g}^k|(e^{t^{\xi_i}}, \theta)}] - \frac{1}{\sqrt{|\tilde{g}^k|(e^t, \theta)}} \partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j} (e^t, \theta) \sqrt{|\tilde{g}^k|(e^t, \theta)}] \right] \partial_{\theta^j} \tilde{w}_i^{\xi_i}(t, \theta),$$

we deduce,

$$A_i \leq C_k |e^{2t} - e^{2t^{\xi_i}}| \left[ |\nabla_{\theta} \tilde{w}_i^{\xi_i}| + |\nabla_{\theta}^2 (\tilde{w}_i^{\xi_i})| \right],$$

If we take  $C = \max\{C_i, 1 \leq i \leq q\}$  and if we use  $(**1)$ , we obtain proposition 4.

We have,

$$c(y_i, t, \theta) = \left( \frac{n-2}{2} \right)^2 + \frac{n-2}{2} \partial_t a + R_g e^{2t}, \quad (\alpha_1)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (\alpha_2)$$

$$c_2 = \left[ \frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}} (\sqrt{b_1}) + |\nabla_{\theta} \log(\sqrt{b_1})|^2 \right], \quad (\alpha_3)$$

Then,

$$\partial_t c(y_i, t, \theta) = \frac{1}{2} \partial_{tt} a + 2e^{2t} R_g(e^t \theta) + e^{3t} < \nabla R_g(e^t \theta) | \theta >,$$

by proposition 1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K_1 e^{2t},$$

Now, we consider the function,  $\bar{w}_i(t, \theta) = \tilde{w}_i(t, \theta) - \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} e^t$ , and  $\lambda > 2 > 0$ .

For  $t \leq t_i = -(2/3) \log u_i(y_i)$ , we have:

$$\begin{aligned} \bar{w}_i(t, \theta) &= e^t \left[ b_1(t, \theta) e^{-t/2} u_i \exp_{y_i}(e^t \theta) - \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} \right] \geq \\ &\geq e^t \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} > 0, \end{aligned}$$

We set,  $\mu_i = \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2}$ .

We use proposition 3 and the same arguments than in [B], we obtain:

**Lemma 4:**

There exists  $\nu < 0$  such that for  $\mu \leq \nu$  :

$$\bar{w}_i^{\mu}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\mu, t_i] \times \mathbb{S}_2,$$

We set,  $\lambda_i = -2 \log u_i(y_i)$ , then,

**Lemma 5:**

$$\bar{w}_i(\lambda_i, \theta) - \bar{w}_i(\lambda_i + 4, \theta) > 0.$$



**Proof of lemma 5:**

Clearly:

$$\bar{w}_i(\lambda_i, \theta) - \bar{w}_i(\lambda_i + 4, \theta) = \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) + \mu_i(e^4 - 1),$$

we deduce lemma 6 from proposition 3.

Let,  $\xi_i = \sup\{\mu \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \forall (t, \theta) \in [\xi_i, t_i] \times \mathbb{S}_2\}$ .

The real  $\xi_i$  exists (see [B]), if we use  $(**2)$ , we have:

$$\tilde{w}_i^{\xi_i}(t, \theta) + |\nabla_{\theta} \tilde{w}_i^{\xi_i}(t, \theta)| + |\nabla_{\theta}^2 \tilde{w}_i^{\xi_i}(t, \theta)| \leq C(R), \forall (t, \theta) \in ]-\infty, \log R] \times \mathbb{S}_2,$$

We can write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) = \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) - \mu_i \bar{Z}_i(e^{2t\xi_i} - e^{2t}),$$

$$-\bar{Z}_i(e^{2t\xi_i} - e^{2t}) = [1 - \frac{1}{4} - \frac{3}{2}\partial_t a - R_g e^{2t} + b_1^{-1/2} b_2 - c_2](e^{2t\xi_i} - e^{2t}) \leq c_1(e^{t\xi_i} - e^t),$$

with  $c_1 > 0$ , car  $|\partial_t a| + |\partial_t b_1| + |\partial_{tt} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C' e^{2t} < 1$ , for  $t$  very small.

We use proposition 4, to obtain on,  $[\xi_i, t_i] \times \mathbb{S}_2$ ,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq c_2[V_i(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + |V_i^{\xi_i} - V_i|(w_i^{\xi_i})^5 + [\mu_i c_1 - C'(R)](e^{t\xi_i} - e^t) \leq 0,$$

Like in [B], after using Hopf maximum principle, we have,

$$\sup_{\theta \in \mathbb{S}_2} [\bar{w}_i^{\xi_i}(t_i, \theta) - \bar{w}_i(t_i, \theta)] = 0.$$

We have:

$$\bar{w}_i^{\xi_i}(t_i, \theta_i) - \bar{w}_i(t_i, \theta_i) = 0, \forall i.$$

We deduce,

$$[u(y_i)]^{1/3} \times \inf_M u_i \leq c, \forall i.$$

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